

**SOLUTION OF TWO-POINT BOUNDARY VALUE PROBLEMS  
USING QUINTIC HERMITE COLLOCATION METHOD**

A

**THESIS**

SUBMITTED TO



**MAHARAJA RANJIT SINGH  
PUNJAB TECHNICAL UNIVERSITY  
BATHINDA, PUNJAB**

IN FULFILMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

**DOCTOR IN PHILOSOPHY**

IN

**MATHEMATICS**

By

SATINDER PAL KAUR  
Regd. No: 17405FPE02

**DEPARTMENT OF MATHEMATICS  
MAHARAJA RANJIT SINGH PUNJAB TECHNICAL UNIVERSITY  
BATHINDA, PUNJAB**

2023

## CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the thesis, entitled “Solution of two-point boundary value problems using quintic Hermite collocation method” in fulfilment of the requirements for the award of the degree of Doctor of Philosophy in Faculty of Sciences, Department of Mathematics and submitted in Maharaja Ranjit Singh Punjab Technical University, Bathinda is an authentic record of my work carried out during a period from July 2017 to November 2022 under the supervision of **Dr. Ajay Kumar Mittal**, Department of Mathematics, Aryabhata Group of Institutes (AGI), Barnala, Punjab and **Dr. Vijay Kumar Kukreja**, Department of Mathematics, SLIET, Longowal, Punjab. The matter embodied in this thesis has not been submitted by me for the award of any other degree of this or any other University/Institute.

**(Satinder Pal Kaur)**

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

**(Dr. AJAY KUMAR MITTAL)**  
Supervisor  
Department of Mathematics  
AGI, Barnala, Punjab

**(Dr. VIJAY KUMAR KUKREJA)**  
Co-supervisor  
Department of Mathematics  
SLIET, Longowal, Punjab

The Ph.D. Viva-Voce examination of \_\_\_\_\_ Research Scholar, has been held on \_\_\_\_\_.

\_\_\_\_\_  
**Supervisor**

\_\_\_\_\_  
**Co-Supervisor**

\_\_\_\_\_  
**External Examiner**

## ACKNOWLEDGEMENT

---

First of all, I would like to thank the Almighty GOD for giving me the strength, knowledge, ability, and opportunity to undertake this research study and complete it satisfactorily.

With sincere gratitude, I acknowledge my indebtedness to my Ph.D. Supervisor, Dr. Ajay Kumar Mittal for familiarizing me with this exciting field of Mathematical Modeling. I would like to thank Dr. Mittal for his invaluable guidance, encouragement, and support in improving my research work. I am grateful and cannot forget the contribution of my Co-supervisor, Dr. Vijay Kumar Kukreja to my research. I learned, not only the knowledge of the subject but also the rigorous scientific approach and the dedicated spirit for work from Dr. Kukreja. I could not have imagined having better advisors and guide for my Ph.D. research study.

I also acknowledge the help extended by the officials of Satia Paper Mills, Rupana (Sri Muktsar Sahib) for providing me with the required industrial data. Otherwise, it would not have been possible to achieve one of the research objectives. Further, I am so appreciative of the Department of Mathematics, MRSPTU, Bathinda because, without their help, I couldn't have finished my study so smoothly.

The completion of research work is never being a one-man show but the collective efforts of all the well-wishers. I offer my sincere thanks to Dr. P. Singh & Dr. N. Parmasur (University of KwaZulu-Natal, Durban, South Africa), Dr. Satvir Singh (IKGPTU, Kapurthala), and Dr. Vikram Mutneja (SBSSU, Ferozepur) as my mentors who were instrumental to me and always encouraged to complete this work. Besides, I want to pay my gratitude to Principal, Guru Nanak College, Sri Muktsar Sahib for motivating me to complete this task.

I am very much thankful to all my friends for giving me support to complete the work. This project could never become a reality without the kind blessings of my mother, mother-in-law, and father-in-law. I owe my deepest gratitude to my husband for his support and understanding in achieving my aspirations. I am also thankful to my loving son Sehajpal Singh and daughter Harnoor Kaur who sacrificed their wishes and maintained whole-hearted patience during this period.

## LIST OF FIGURES

S.No.	Figure number	Title	Page No.
1.	1.1	A schematic modelling process	4
2.	5.1	Stability region when eigenvalues are complex	70
3.	5.2	Eigenvalues of matrix A for linear model-1&2 with N=51, $\Delta t=0.0001$	70
4.	5.3	Eigenvalues of matrix A for linear model-3 with N=51, $\Delta t=0.0001$	71
5.	5.4	Eigenvalues of matrix A for linear model-4 with N=51, $\Delta t=0.0001$	71
6.	5.5	Eigenvalues of matrix A for nonlinear model-1&2 with N=51, $\Delta t=0.0001$	72
7.	5.6	Eigenvalues of matrix A for nonlinear model-3 with N=51, $\Delta t=0.0001$	72
8.	5.7	Matrix structure for the system of equations obtained using QHCM	74
9.	6.1	Comparison of the exact solution and QHCM for 30 elements	81
10.	6.2	Comparison of absolute error obtained using QHCM with CHCM for $Pe=40$	81
11.	6.3	Comparison of exit solute concentration for $Pe=40$ by dividing the domain into the various number of elements	83
12.	6.4	Relative error comparison between QHCM and CHCM for $Pe=1$ , $M=10$	83

13.	6.5	Relative error comparison between QHCM and CHCM for $Pe=10$ , $M=40$	84
14.	6.6	Relative error comparison between QHCM and CHCM for $Pe=40$ and $M=40$	84
15.	6.7	Relative error comparison of QHCM with OCFE and CHCM for $Pe=80$	87
16.	6.8	Comparison of relative error by division of domain into different elements for $Pe=40$	88
17.	6.9	Relative Error comparison for $Pe=20$ , $Bi=5$ , $\mu=0.033$	95
18.	6.10	Comparison of CSCM, CHCM and QHCM for $Pe=10$ , $Bi=1.5$ , $\mu=0.0142$	95
19.	6.11	Effect of different values of $Pe$ and $Bi$ on the concentration of solute at exit level	97
20.	6.12	Effect on exit solute concentration for different values of $R_d$	97
21.	6.13	Effect on exit solute concentration for large values of $R_d$	99
22.	6.14	For $R_d=1$ , the effect of different values of $Pe$ on the solute concentration	99
23.	6.15	Exit solute concentration profile for $Pe=40$ for division of domain into different elements	107
24.	6.16	Influence of small value of $Pe$ on the concentration profile	111
25.	6.17	Influence of large value of $Pe$ on the concentration profile	111

26.	6.18	Influence of $Pe$ on exit solute concentration	112
27.	6.19	Influence of $D_L$ on concentration profiles for $u=4.0E-03$ and $L=0.02$	112
28.	6.20	Influence of $u$ on concentration profile for $L=0.02$ and $D_L=5.9E-06$	113
29.	6.21	Influence of $L$ on concentration profiles for $u=4.0E-03$ and $D_L=5.9E-06$	114
30.	6.22	Influence of porosity on concentration profile	115
31.	6.23	Surface plot for non-linear model for $Pe=1$	116
32.	6.24	Surface plot for non-linear model for $Pe=10$	116
33.	6.25	Surface plot for non-linear model for $Pe=40$	117
34.	6.26	Surface plot for the non-linear model for $Pe=300$	117
35.	6.27	Influence of $Pe$ on displacement ratio	121
36.	6.28	Influence of $Pe$ on bed efficiency	122
37.	6.29	3D plot for concentration profile at exit level using experimental data of Grahs (1974)	122
38.	6.30	3D plot for concentration profile at exit level using experimental data of Kukreja (1996)	123
39.	6.31	3D plot for concentration profile at exit level using experimental data of Arora & Potuček (2012)	123
40.	6.32	3D plot for concentration profile at exit level for $Pe=1$	126
41.	6.33	3D plot for concentration profile at exit level for $Pe=10$	126
42.	6.34	3D plot for concentration profile at exit level for $Pe=40$	127
43.	6.35	3D plot for concentration profile at exit level for $Pe=300$	127

44.	6.36	Comparison of concentration profile at exit level for nonlinear model-1 and nonlinear model-2	128
45.	6.37	Comparison of exit solute concentration profile by division of domain into different elements for $Pe=10$ , 40, 100	129
46.	6.38	Comparison of exit solute concentration profile for different values of $Pe$	130
47.	6.39	3D plot for concentration profile at exit level for $Pe=1$	130
48.	6.40	3D plot for concentration profile at exit level for $Pe=10$	131
49.	6.41	3D plot for concentration profile at exit level for $Pe=40$	131
50.	6.42	3D plot for concentration profile at exit level for $Pe=300$	132
51.	6.43	Comparison of OCFE and QHCM in terms of absolute error	136
52.	6.44	Influence of $Pe$ on the concentration of solute at exit level	138
53.	6.45	Influence of distribution ratio on the concentration of solute at exit level	138
54.	6.46	Effect of different washing run of Table 6.22 on solution profiles	139
55.	6.47	Effect of bed porosity on solution profiles	140
56.	7.1	Satia Paper Mills	142
57.	7.2	Brown stock washer	143
58.	7.3	Effect of $Pe$ for $c_0=0.08$ , $A=0.01195$ , $B=2.708$ , $c_s=0.01$ , $C_F=55.88$ and $e=0.962$	147

59.	7.4	Effect of porosity ( $\epsilon$ ) for $c_0 = 0.08$ , $A = 0.01195$ , $B = 2.708$ , $c_s = 0.01$ , $C_F = 55.88$	147
60.	7.5	Effect of fiber consistency for $c_0 = 0.08$ , $A = 0.01195$ , $B = 2.708$ , $c_s = 0.01$ , $e = 0.961$	148
61.	7.6	Effect of $Pe$ for $c_0 = 0.08$ , $A = 0.01195$ , $B = 2.708$ , $c_s = 0.01$ , $C_F = 86.76$	149
62.	7.7	Effect of porosity for $c_0 = 0.08$ , $A = 0.01195$ , $B = 2.708$ , $c_s = 0.01$ , $C_F = 8$	150



## LIST OF TABLES

S.No.	Table number	Title	Page No.
1.	6.1	Comparison of exit solute concentration w.r.t. exact and numerical solutions	80
2.	6.2	Comparison of $\ L\ _{\infty}$ norm CHCM and QHCM	86
3.	6.3	Comparison of $\ L\ _2$ norm CHCM and QHCM	86
4.	6.4	Comparison between the exact solution and numerical results derived using QHCM	87
5.	6.5	Results for supremum norms ( $\ L\ _{\infty}$ ), Euclidean norms ( $\ L\ _2$ ) and CPU time	89
6.	6.6	Comparison of maximum absolute error for CHCM and QHCM	90
7.	6.7	Comparison of $\ L\ _2$ (Euclidean) norm for CHCM and QHCM	90
8.	6.8	Comparison of numerical results with exact for different $Pe$ and $Bi$	93
9.	6.9	Comparison of numerical results of QHCM with CHCM	94
10.	6.10	Comparison of results of $\ L\ _2$ and $\ L\ _{\infty}$ norms	94
11.	6.11	Comparison in total elapsed time (in sec) between QHCM and CSCM for $Pe=60$ , $Bi=7.5$ , and $\mu=0.033$	94
12.	6.12	Experimental data of washing cell of Grähs (1974)	101
13.	6.13	Washing cell experimental data of Kukreja (1996)	101
14.	6.14	Experimental data of washing cell (Arora & Potuček, 2012)	101

15.	6.15	Numerical results of concentration of solute at exit level using QHCM for nonlinear model with different range of $Pe$ & $M$	103
16.	6.16	Comparison of solution profile using QHCM and CHCM with $C_0 = 0.278; A_0 = 0.01195; B_0 = 2.708; C_s = 0.000; e = 0.942; C_F = 106.9$	104
17.	6.17	Comparison of solution profile using QHCM and CHCM with $C_0 = 8.33; A_0 = 0.00052; B_0 = 0.000625; C_s = 0.005; e = 0.96; C_F = 66.17$	105
18.	6.18	CPU time (in a sec) for the concentration of solute by dividing the domain into different elements	106
19.	6.19	Comparison of rate of convergence of CHCM and QHCM	107
20.	6.20	Solution profile for nonlinear model-2 for various values of $Pe$	125
21.	6.21	Comparison between experimental and numerical values with published results	135
22.	6.22	Experimental data for different washing runs by Arora & Potůček (2012)	139
23.	7.1	Experimental data collected from the paper mill	144
24.	7.2	Expression used to calculate data from Kukreja (1996)	144
25.	7.3	Calculated data for simulation	145